MATH 4000-PROBLEM SOLVING FOR PUTNAM, FALL 2019
HOMEWORK NO. 3

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Problem 1. Let $(G, \cdot)$ be a group such that $(ab)^2 = a^2b^2$, for all $a, b \in G$. Show that G is an abelian group.

classic

Problem 2. Let R be a ring with identity with the property that $(xy)^2 = x^2y^2$ for all $x, y \in R$. Show that R is commutative.

some old Romanian olympiad

Problem 3. Show that a finite group cannot be the union of two of its proper subgroups. Does the statement remain true if "two" is replaced by "three"?

Putnam B2, 1969

Problem 4. Let $A \in M_2(\mathbb{R})$ such that $\det A = -1$. Show that $\det(A^2 + I_2) \geq 4$.

RNMO SL, 2003

Problem 5. Let $A, B \in M_2(\mathbb{R})$. Show that

$$\det(A^2 + B^2) \geq \det(AB - BA).$$

RMO, 2003

Problem 6. Let $A \in M_3(\mathbb{C})$ such that $\operatorname{tr}(A) = \operatorname{tr}(A^2) = 0$. Show that if $|\det A| < 1$, then the matrix $I_3 + A^n$ is invertible for all $n \geq 1$.

RMO, 2002

Problem 7. Let $A, B \in M_3(\mathbb{C})$ such that $(AB)^2 = A^2B^2$ and $(BA)^2 = B^2A^2$. Prove that $(AB - BA)^3 = O_3$.

RNMO SL, 2014

Problem 8. Are there any matrices $A, B \in M_3(\mathbb{C})$ such that $(AB - BA)^2 = I_3$?

Problem 9. For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I_2$, where

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

Show that $\lim_{n\to\infty} d_n = \infty$.

Putnam B4, 1994

Problem 10. Let A, B be matrices of size 3×2 and 2×3 respectively. Suppose that their product in the order AB is given by

$$AB = \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}$$

Show that the product BA is given by

$$BA = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

Putnam B1, 1969

Problem 11. Let A be an $m \times n$ matrix with rational entries. Suppose that there are at least $m + n$ distinct prime numbers among the absolute values of the entries A. Show that the rank of A is at least 2.

Putnam B3, 2014

Problem 12. Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i, j)}$$

for $1 \le i, j \le n$. Compute $\det(A)$.

Putnam A2, 2014

Problem 13. Let A, B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, then can $A^2 + B^2$ be invertible?

Putnam A2, 1991

Problem 14. A and B are square complex matrices of the same size and

$$\text{rank}(AB - BA) = 1.$$

Show that $(AB - BA)^2 = O_n$.

IMC, 2000

Problem 15. Let M be a real $n \times n$ matrix with all entries in $\{0, 1\}$ such that all eigenvalues are positive. Prove that all eigenvalues are equal to 1.

Problem 16. Let $A \in M_n(\mathbb{R})$ such that $|\operatorname{tr}(A)| > n$. Show that $A^k \neq A^p$ for all k, p positive integers with $k \neq p$.

Putnam B3, 2001

Problem 17. Let $A \in M_n(\mathbb{C})$ such that $\operatorname{tr}(A^k) = 0$ for all $1 \leq k \leq n-1$, and $\operatorname{tr}(A^n) = n$. Show that $A^n = I_n$.

classic

Problem 18. Let $A \in M_n(\mathbb{R})$ such that $A^3 = A + I_n$. Show that $\det(A) > 0$.

classic

Problem 19. Let $N \in M_n(\mathbb{C})$ be a *nilpotent* matrix. Show that

(i) $\det(I_n + N) = 1$.
(ii) $\det(A + N) = \det(A)$, for any matrix $A \in M_n(\mathbb{C})$ such that $AN = NA$.

IMC, 1998

Problem 20. . Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a symmetric $n \times n$ matrix with real entries and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

IMC, 2014