MATH 4000-PROBLEM SOLVING FOR PUTNAM, FALL 2019
HOMEWORK NO. 2

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Problem 1. Let $x_1, x_2, \ldots, x_7$ be real numbers. Show that there exists $i, j = 1, 2, \ldots, 7$ distinct such that

$$\left|\frac{x_i - x_j}{1 + x_i x_j}\right| \le \frac{1}{\sqrt{3}}.$$

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Problem 2. Let there be given 9 lattice points (points with integral coordinates) in 3-dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points.

Putnam A1, 1971

Problem 3. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

Putnam A1, 1978

Problem 4. (a) Prove that there exist integers a, b, c, not all zero and each of absolute value less than one million, such that

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.$$

(b) Let a, b, c, not all zero and each of absolute value less than one million. Prove that

$$|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}.$$

Putnam A4, 1980

Problem 5. Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left|m + \sum_{s \in S} s\right| \le \frac{1}{n+1}$$

Putnam B2, 2006

Problem 6. Let $d_1, d_2, \dots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

Putnam A1, 2012

Problem 7. Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Putnam A1, 2013

Problem 8. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

Putnam B2, 2000

Problem 9. Let n be a positive integer such that $n+1$ is divisible by 24. Prove that the sum of all the divisors of n is divisible by 24.

Putnam B1, 1969

Problem 10. Prove that

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p},$$

for all integers p, a, b, with p prime, $p > 0$, and $a \geq b \geq 0$.

Putnam A5, 1977

Problem 11. Let $a_0 = 1, a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

Putnam A2, 2015

Problem 12. Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.

Putnam A5, 1972

Problem 13. Let p be a prime greater than 3. Prove that

$$p^2 \Big| \sum_{i=1}^{\lfloor \frac{2p}{3} \rfloor} \binom{p}{i}.$$

Putnam A5, 1996

Problem 14. Let p be an odd prime. Show that the equation $x^2 \equiv -1 (\operatorname{mod} p)$ has a solution if and only if $p \equiv 1 (\operatorname{mod} 4)$.

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Problem 15. For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate:

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}$$

Putnam B3, 2001

Problem 16. Define a positive integer n to be *squarish* if either n is itself a perfect square or the distance from n to the nearest perfect square is a perfect square. For example, 2016 is squarish, because the nearest perfect square to 2016 is $45^2 = 2025$ and $2025 - 2016 = 9$ is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squarish.)

For a positive integer N, let $S(N)$ be the number of squarish integers between 1 and N, inclusive. Find positive constants α and β such that

$$\lim_{N \to \infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist.

Putnam B2, 2016