MATH 4000-PROBLEM SOLVING FOR PUTNAM, FALL 2019
HOMEWORK NO. 1

LECTURER: CEZAR LUPU

Problem 1. Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)},$$

for $x>0$.

Putnam B1, 1998

Problem 2. Let $a,b,c>0$. Show the following inequalities

$$(a+b)(b+c)(c+a)\geq 8abc,$$

$$(a+b)(b+c)(c+a)\geq \frac{8}{9}(a+b+c)(ab+bc+ca),$$

$$a^3+b^3+c^3\geq \frac{(a+b+c)^3}{9},$$

and

$$3(a^3+b^3+c^3)\geq (a+b+c)(a^2+b^2+c^2).$$

Problem 3. Let $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ be nonnegative numbers. Show that

$$\sqrt[n]{a_1a_2\ldots a_n}+\sqrt[n]{b_1b_2\ldots b_n}\leq \sqrt[n]{(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)}.$$

Putnam A2, 2003

Problem 4. Let $x_1,x_2,\ldots,x_n>0$ and $\alpha_1>0,\alpha_2>0,\ldots,\alpha_n>0$. Show that

$$\frac{x_1^2}{\alpha_1}+\frac{x_2^2}{\alpha_2}+\ldots+\frac{x_n^2}{\alpha_n}\geq \frac{(x_1+x_2+\ldots+x_n)^2}{\alpha_1+\alpha_2+\ldots+\alpha_n}.$$

special case of the Cauchy-Schwarz's inequality

Application. Show that for any positive reals a,b,c we have

$$\sum_{cyc}\frac{a}{b+c}\geq \frac{3}{2}.$$

Problem 5. Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1+\cdots+k_n = 5n-4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Putnam B2, 2005

Problem 6. Let $z = x+y\cdot i$ be a complex number with x, y rational numbers and with $|z| = 1$. Show that the number $|z^{2n} - 1|$ is rational for every positive integer n.

Putnam B2, 1973

Problem 7. Suppose that $a_1, a_2, \ldots, a_n$ are real $(n > 1)$ and

$$A + \sum_{i=1}^{n} a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Prove that $A < 2a_i a_j$ for $1 \leq i < j \leq n$.

Putnam B5, 1977

Problem 8. Let m, n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

Putnam B2, 2004

Problem 9. For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1, c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Putnam B1, 2013

Problem 10. Show that if $H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, then

$$n(n+1)^{1/n} < n + H_n, n > 1,$$

and

$$(n-1)n^{-1/(n-1)} < n - H_n, n > 2.$$

Putnam B6, 1975

Problem 11. Let n be a positive integer, and let

$$f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}.$$

Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Putnam B2, 2018

Problem 12. Prove that a polynomial with only real roots and all coefficients equal to ± 1 has degree at most 3.

Inspired by Putnam B6, 1968

Problem 13. Determine all polynomials $P(x)$ such that $P(x^2+1) = (P(x))^2+1$, and $P(0) = 0$.

Putnam A2, 1971

Problem 14. The three vertices of a triangle of sides a, b and c are lattice points and lie on a circle of radius R. Show that $abc \geq 2R$. (Lattice points ar epoints in the Euclidian plane with integral coordinates.)

Putnam A3, 1971

Problem 15. A quadrilateral which can be inscribed in a circle is said to be *cyclic*. A quadrilateral which can be circumscribed to a circle is said to be *circumscribable*. Show that if a circumscribable quadrilateral of sides a, b, c, d has area $A = \sqrt{abcd}$, then it is also cyclic.

Putnam B6, 1970

Problem 16. Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB.

Putnam A1, 2015